RADIATIVE TRANSFER OF LOCALIZED WAVES

A local diffusion theory

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1. Introduction

Almost twenty years ago, after one century of radiative transfer, and twenty years after the first paper by Anderson on localization [1], the first publications of the self-consistent theory of localization appeared. Following ideas of Götze [2], Vollhardt and Wölfle demonstrated in a series of pioneering papers [3-5] the importance of the so-called "most-crossed" diagrams for the renormalization of the diffusion coefficient in wave diffusion. These diagrams are now known to be at the very base of all kinds of weak localization phenomena, such as the Sharvin-Sharvin effect [6] and enhanced backscattering [7]. Twenty years ago it became very urgent to understand the role of wave localization in a context that concerns *transport* of waves in *open* media, and to include interference effects into transport theory. This goal was, and still is very ambitious. Earlier studies had shown localization to be a non-perturbational phenomenon [8]. Very few people believed that localization could once be understood by generalizing an ordinary, classical Boltzmann equation. At the same time, however, researchers were eager to give "great" principles, such as the Thouless criterion [9], and finite-size

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scaling [10] a more microscopic base. This requires an understanding of localization in open media.

Anno 2000, random-matrix theory [11] and the self-consistent theory of localization have provided us a wealth of information how transport of waves in a disordered medium is affected by the nearness of localization. Such studies were particularly stimulated by experiments with classical waves, such as microwaves [12-14], visible and infrared light [15, 16, 17] and acoustic waves [18], whose natural language had always been radiative transfer. To generalize classical transport for interferences, one has the choice of either solving simplified models exactly, or to find approximate solutions of "exact" models.

Random-matrix theory (RMT) is of the first kind. Based on the elegant chaos theory by Dyson and others, RMT is now able to predict many features of wave diffusion, including fluctuations, even in the localized regime [19], and even in a non-perturbational way. Two aspects are still unsolved. First, RMT was constructed for quasi-one dimensional systems, for which no mobility edge is believed to exist, and a generalization to higher dimensional systems does not seem feasible. This eliminates RMT as a candidate to understand features like enhanced backscattering or anomalous transmission. Second, standard RMT is basically a stationary theory. Only very recently, Beenakker *et al.* published a dynamic variant of RMT in reflection [20].

The strong point of the self-consistent theory is that it is based on a *rigorous* argument - reciprocity - applied to an "exact" equation - the Bethe-Salpter equation - that applies in any random system. A weak point is that the theory can only be worked out approximately, for instance, by employing the diffusion approximation and a low-disorder expansion. Unfortunately, this weak point is often misused as an argument against the whole principle. The approximate theory gives critical exponents around the transition, which are typical of mean field approximations and do not agree with *ab-initio* studies of the Anderson Tight Binding model [21]. In addition, the theory only applies to field-field correlation functions, and does not do any predictions about fluctuations of e.g., conductance. Recent experimental work by Genack etal. [17] has demonstrated the importance of fluctuations to get a complete view on Anderson localization. Finally, the self-consistent theory seems to fail in the absence of time-reversal symmetry, e.g., when a magnetic field is present in the medium [5], probably because this violates one of the two basic assumptions of the theory. Despite the above weak points, the self-consistent theory has given us deeper insight into the role of Anderson localization in wave transport. It has confirmed the Thouless criterion as a universal criterion for wave localization in open media and has put forward the Ioffe-Regel criterion, anticipated already in the early sixties [22], as a criterion for localization in infinite 3D media. As was shown and emphasized by Wölfle and Vollhardt [5], the self-consistent theory agrees in great detail with scaling arguments for the dynamic diffusivity $D(\Omega)$ [23] and with the scaling theory for the DC conductance [10].

One pertinent controversy initiated by the self-consistent theory concerns the scale-dependence of the diffusion kernel itself. Scaling theory has led to a homogeneous but "scale-dependent" diffusivity kernel $D(\Omega, \mathbf{r}-\mathbf{r}')$, with Fourier transform $D(\Omega, \mathbf{q})$. Near the mobility edge one has suggested $D(\Omega, q) \sim q$ [24-26]. The absence of such q-dependence in the self-consistent theory is sometimes considered as a serious failure, in spite of its other successes mentioned above. Recently, we presented a local formulation of the self-consistent theory [27] that will be explained below. We find that weak localization effects lower the diffusion coefficient, but we also infer that the suppression is different near the boundaries. As a result, we encounter a new feature: a spatially inhomogeneous, but *local* diffusion coefficient $D(\mathbf{r})$. At the mobility edge, our local formulation predicts a scale-dependence $D(z) \sim 1/z$ of the diffusion coefficient of a slab geometry, leading to a transmission $T \sim 1/L^2$ of a slab with thickness L, and a rounding of line shape in enhanced backscattering. Both properties have been observed [15, 16], but were previously interpreted in terms of a scale-dependent diffusivity $D(q) \sim q$ [26]. As a bonus, our local variant of the self-consistent theory is able to deal explicitly with boundary conditions in an almost conventional way. This facilitates "engineering" with the self-consistent theory.

2. Basic Elements of the Selfconsistent Theory

The Green's function $G(\mathbf{r_1}, \mathbf{t_1} \to \mathbf{r_2}, \mathbf{t_2})$ describes the propagation of a wave at position $\mathbf{r_1}$ at time t_1 to position $\mathbf{r_2}$ at time t_2 and can be constructed from the underlying wave equation. We denote its ensemble average by $\langle G \rangle$. In its most general form, transport theory is a theory for the ensemble-averaged "two-particle" Green's function

$$\langle G(\mathbf{r_1}, \mathbf{t_1} \to \mathbf{r_3}, \mathbf{t_3})\mathbf{G}^*(\mathbf{r_2}, \mathbf{t_2} \to \mathbf{r_4}, \mathbf{t_4})\rangle \equiv \Gamma(\mathbf{r_1}, \mathbf{t_1}, \mathbf{r_2}, \mathbf{t_2}, \mathbf{r_3}, \mathbf{t_3}, \mathbf{r_4}, \mathbf{t_4}) \,. \tag{1}$$

This object is related to the total intensity of the radiation field. It relates intensity properties at the "source" (indices 1 and 2) to the ones at the "detector" (indices 3 and 4). In a linear random medium, an object $U(\mathbf{r_1}, \mathbf{t_1} \cdots \mathbf{r_4}, \mathbf{t_4})$ should exist that generates the two-particle Green's function in the following symbolical way,

$$\Gamma = U + U \cdot \langle G \rangle \times \langle G^* \rangle \cdot \Gamma \equiv U + R \,. \tag{2}$$

The new object U is called the *irreducible vertex*. The dots denote convolutions in space-time. This so-called Bethe-Salpeter equation can be formally iterated to yield a multiple scattering series for the two-particle Green's function with U as an elementary building block. This iteration provides a new "reducible vertex" R that contains all multiple scattering events except the elementary block U.

Although U may look like a complicated object, it has one simple, but important property that we will now discuss. Let us for simplicity adopt monochromatic waves with frequency ω so that the time dependence $\exp(-i\omega t_i)$ becomes trivial. We first observe that spatial reciprocity (i.e., interchanging detector and source) imposes the following reciprocity relations for the two-particle Green's function [5, 28],

$$\Gamma(\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r_4}) = \mathbf{\Gamma}(\mathbf{r_3},\mathbf{r_4},\mathbf{r_1},\mathbf{r_2}) = \mathbf{\Gamma}(\mathbf{r_3},\mathbf{r_2},\mathbf{r_1},\mathbf{r_4})\,. \tag{3}$$

The first identity is the well-known, classical reciprocity relation in radiative transfer [29], and is also obeyed by R and U separately. However, neither U nor R is expected to obey the second reciprocity identity. A somewhat technical inspection learns that interchanging source and detector *only* for the field going $1 \to 3$ without doing the same procedure for the field that travel from $2 \to 4$ turns *any* contribution to R into an irreducible contribution to U [30]. The reverse, however, is not true. This is expressed by the following, unique decomposition

$$U(\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r_4}) = \mathbf{C}(\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r_4}) + \mathbf{S}(\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r_4})\,, \tag{4}$$

where the vertex C is obtained from R using the reciprocity operation,

$$C(\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r_4}) = \mathbf{R}(\mathbf{r_3},\mathbf{r_2},\mathbf{r_1},\mathbf{r_4})\,, \tag{5}$$

and S is a set of scattering diagrams that is transformed into itself,

$$S(\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r_4}) = \mathbf{S}(\mathbf{r_3},\mathbf{r_2},\mathbf{r_1},\mathbf{r_4})\,. \tag{6}$$

Single scattering can easily be seen to be part of S. The classical picture of radiative transfer emerges when for U and S single scattering is adopted as an approximate building block for multiple scattering [31]. However, this procedure disregards the existence of C, so that *classical radiative transfer does not obey the reciprocity principle* (3). The vertices C and S give rise to physical phenomena that are not described by classical radiative transfer. The object C is the most general formulation possible to describe enhanced backscattering, an interference effect that has been observed with light [7] and acoustic waves [32], and recently even in a cold rubidium gaz [33]. In addition to single scattering, the set S contains recurrent scattering events. They are known to affect the celebrated enhancement factor of two in backscattering [28] as also observed [34].

For practical calculations, one can observe that the vertices R and C typically start and end at a scattering particle that is assumed small compared to the mean free path. Therefore, one expects the relations [35]

$$\begin{aligned} R(\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}, \mathbf{r_4}) &\approx \delta(\mathbf{r_1} - \mathbf{r_2})\delta(\mathbf{r_3} - \mathbf{r_4})\mathbf{F}(\mathbf{r_1}, \mathbf{r_3}), \\ C(\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}, \mathbf{r_4}) &\approx \delta(\mathbf{r_3} - \mathbf{r_2})\delta(\mathbf{r_1} - \mathbf{r_4})\mathbf{F}(\mathbf{r_1}, \mathbf{r_3}), \end{aligned} \tag{7}$$

leaving only two independent positions by means of the function $F(\mathbf{r_1}, \mathbf{r_3})$. The reciprocity principle (5) imposes the appearance of the same, symmetric function F in both R and C. Physically, the function F describes how (stationary) multiple scattering transfers radiation from one place to the other. It has a hydrodynamic long range behavior: in an infinite 3D medium it decays as $1/|\mathbf{r_1} - \mathbf{r_3}|$ for large separations. This in sharp contrast to the vertex S, which represents a kind of "super" single scattering, including loops in the medium. One may assert short range behavior, i.e.,

$$S(\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}, \mathbf{r_4}) \approx \delta(\mathbf{r_1} - \mathbf{r_2})\delta(\mathbf{r_3} - \mathbf{r_4})\delta(\mathbf{r_1} - \mathbf{r_3})\mathbf{S}(\mathbf{r_1}), \tag{8}$$

as it should be if S is to be a good "building block."

The relations (2), (4) and (5) show that reciprocity relates the output Γ of the transport equation directly to its input U. As a result, the problem of writing down and solving a transport equation is a *self-consistent* problem. This is the basic message of the self-consistent theory of localization. We refer to the excellent review by Wölfle and Vollhardt [5] for some subtle complications and for more details.

2.1. DIFFUSION APPROXIMATION

The self-consistent problem has so far only been worked out in the diffusion approximation. On length scales larger than the mean free path, the transport equation for $F(\mathbf{r_1}, \mathbf{r_2})$ reduces to a diffusion equation. This is a well-known consequence of flux conservation [31]. The diffusion coefficient is related to the irreducible vertex U [31]. The Boltmann diffusion coefficient D_B is obtained if for U ordinary single scattering is adopted. Inclusion of C gives the following equation,

$$\frac{1}{D(\mathbf{r})} = \frac{1}{D_B} + \frac{F(\mathbf{r}, \mathbf{r})}{\pi \mathbf{v_E}(\mathbf{k})\rho(\mathbf{k})} . \tag{9}$$

The physics behind the "weak localization" term is the constructive interference of reciprocal paths at position $\mathbf{r}$, expressed by the "return probability" $F(\mathbf{r}, \mathbf{r})$. We will ignore the difference between extinction length, scattering and Boltzmann transport mean free path and represent all by ℓ. With v_E the transport speed of light and k its wave number we have (in 3D) the familiar relations $D_B = \frac{1}{3}v_E\ell$ [31], and $\rho(k) \approx k^2/\pi^2 v_E$ for the density of states per unit volume. Both k, ℓ and v_E have been calculated near the localization threshold [36].

The stationary diffusion equation for F is

$$-\nabla \cdot D(\mathbf{r})\nabla \mathbf{F}(\mathbf{r}, \mathbf{r}') = \frac{\mathbf{4}\pi}{\ell}\delta(\mathbf{r} - \mathbf{r}')\,. \tag{10}$$

The factor $4\pi/\ell$ appears when single scattering is adopted as a source for multiple scattering.

Equations (9) and (10) must contain *one and the same* diffusion coefficient, and one seeks for a "self-consistent" solution. In infinite media, $F(\mathbf{r}, \mathbf{r}')$ is translationally invariant, so that the return probability $F(\mathbf{r}, \mathbf{r})$ and diffusion coefficient $D(\mathbf{r})$ do not depend on $\mathbf{r}$. In reciprocal space is $C(q) = 4\pi/\ell D q^2$, so that $C(\mathbf{r}, \mathbf{r}) = \sum_{\mathbf{q}} \mathbf{C}(\mathbf{q}) \sim \mu/\mathbf{D}\ell^{\mathbf{2}}$, assuming an upper cut-off $q_{\max} = \mu/\ell$ to regularize the diverging wavenumber integral. Hence,

$$D = D_B\left(1 - \frac{\mu}{k^2\ell^2}\right)\,. \tag{11}$$

This is the standard "Vollhardt and Wölfle" result in three dimensions. The mobility edge, if defined by $D = 0$ [8], obeys a Ioffe-Regel type criterion as derived microscopically by John *et al.* [37] and Economou *et al.* [38], and agrees with numerical studies of the Anderson Tight Binding model [39, 40]. The obligation to use a cut-off, which somewhat arbitrarily eliminates wave paths shorter than the mean free path from the return probability, highlights the partial failure of the diffusion approximation. Some controversy existed about the choice of this cut-off, an important issue as it influences the exact location of the mobility edge [38, 41, 42]. A careful analysis of the exact formulas demonstrated the choice $q_{\max} = \mu/\ell$ to be approximately correct [43]. For $\mu = 1$, the mobility edge is conveniently located at $k\ell = 1$, which is close to predictions made by the Potential Well Analogy ($k\ell = 0.84$) [38] and by point scatterer models ($k\ell = 0.97$) [43].

The essence of our work is that absence of translational symmetry in finite media imposes the diffusion coefficient $D(\mathbf{r})$ to depend on $\mathbf{r}$. This conclusion is unavoidable if one doesn't wish to give up the basic ingredients of the self-consistent theory of localization: reciprocity and flux conservation. Previous applications of the self-consistent theory accounted for the boundaries by means of a second, lower cut-off [3, 4].

3. Application to a Slab Geometry

We consider stationary propagation in a slab geometry of thickness L and infinite width, and Fourier transform ($\mathbf{q}_{\|}$) the transverse coordinate. For $0 < z < L$, Eqs. (9) and (10) become

$$-\partial_z D(z)\, \partial_z\, F(z, z', q_{\|}) + D(z) q_{\|}^2 F(z, z', q_{\|}) = \frac{4\pi}{\ell} \delta(z - z')\,, \quad (12)$$

$$\frac{1}{D(z)} = \frac{1}{D_B} + \frac{2}{k^2 \ell} \int_0^{1/\ell} \mathrm{dq_{\|}\, q_{\|} F(z, z, q_{\|})}\,, \quad (13)$$

$$F(0, z', q_{\|}) - z_e(0) \partial_z F(0, z', q_{\|}) = 0\,, \quad (14)$$

$$F(L, z', q_{\|}) + z_e(L) \partial_z F(L, z', q_{\|}) = 0\,. \quad (15)$$

The last two equations are the familiar radiative boundary conditions at both sides of the slab, featuring the "extrapolation lengths" $z_e(0/L) \equiv 3 z_0 D(0/L)/v_E$ [44]. They contain the diffusion coefficient, $D(0/L)$, at the boundaries so that z_e is always non-zero, even in the localized regime, when D vanishes in the bulk. The value, $z_0 = \frac{2}{3}$, corresponds to no internal reflection, but is much larger in recent localization experiments [15, 16] due to internal reflection. Equation (12) is recognized as an ordinary, second order differential equation with a source term. Without the latter, two independent solutions $f_{\pm}(z)$ exist with a non-zero Wronskian $W(q_{\|}) \equiv D(z) \times (f'_+ f_- - f'_- f_+)$, independent of z.

We first discuss the semi-infinite medium $L = \infty$. Let $f_+(z)$ be the growing solution. As $F(z, z', q_{\|})$ must be bounded at large z, z', Eq. (12) is solved for

$$F(z, z', q_{\|}) = \frac{f_+(z_<) f_-(z_>)}{W(q_{\|}) \ell / 4\pi} - P(q_{\|}) f_-(z) f_-(z')\,, \quad (16)$$

where $z_< = \mathrm{min(z, z')}$, $z_> = \mathrm{max(z, z')}$ and $P(q_{\|})$ follows easily from the boundary condition (14). At the mobility edge we assert the simple algebraic form,

$$D(z) = \frac{D(0)}{1 + z/\xi_c}\,, \quad (17)$$

with two free parameters $D(0)$ and ξ_c. The homogeneous solutions would then be,

$$f_+(z) = (z + \xi_c) I_1 \left(q_{\|} [z + \xi_c] \right)\,,$$
$$f_-(z) = (z + \xi_c) K_1 \left(q_{\|} [z + \xi_c] \right)\,, \quad (18)$$

in terms of the modified Bessel functions I_1 and K_1 with Wronskian $W = D(0)\xi_c$ [45]. Equation (16) learns that $C(z, z, q_{\|})$ rises linearly in z for large z and that Eq. (13) is indeed asymptotically satisfied. Equation (13) evaluated at $z = 0$ gives a relation between $D(0)$ and ξ_c. The remaining freedom in ξ_c can be used to optimize self-consistency below 0.05 %. Both ξ_c and $D(0)$ depend heavily on the parameter z_0 in the boundary condition (see Table 3).

Table 1. Solution $D(z) = D(0)/(1 + z/\xi_c)$ of the self-consistent equations at the mobility edge $k\ell = 1$ for a semi-infinite slab as a function of the parameter z_0 that controls internal reflection at the boundary. The middle column reveals that $D(0) \sim 1/z_0$. For $z_0 = \infty$ one expects $D = 0$ for all z.

z_0	$D(0)/D_B$	ξ_c/ℓ
2/3	0.642	1.5
3	0.336	3
5	0.249	4
7	0.203	6
10	0.159	8
20	0.0968	25

The line shape $I_c(\theta)$ in enhanced backscattering can be obtained from $C(z, z', q_\|)$ using standard methods [46]. Insight is provided by the approximate formula $I_c(\theta) \approx C(z = \ell, z' = \ell, q_\| = 2k \sin\theta/2)$, used by De Vries *et al.* [44]. The line shape exhibits a logarithmic rounding

$$I_c(\theta) \sim 1 + z_e(0)\xi_c q_\|^2 \log(q_\|\xi_c)\,, \tag{19}$$

when $q_\|\xi \ll 1$, rather than the familiar cusp $I_c(\theta) \sim 1 - z_e|q_\||$ [46]. Berkovits and Kaveh [26] predicted a rounding of the line shape on the basis of the non-local diffusion kernel $D(q)$.

The localized regime corresponds to $k\ell < 1$. We may assert the solution $D(z) = D(0)\exp(-2z/\xi)$, with ξ the localization length. We find $f_\pm(z) = \exp(-\lambda_\pm z)$ with $\lambda_\pm = 1/\xi \pm \sqrt{q_\|^2 + 1/\xi^2}$, and Wronskian $W = 2D(0)\sqrt{q_\|^2 + 1/\xi^2}$. Equation (13) is indeed satisfied for $z \gg \xi$ if we adopt the localization length $\xi/\ell = 2(k\ell)^2/[1 - (k\ell)^4]$, with a critical exponent of unity. The same equation evaluated at $z = 0$ provides $D(0)$. The above exponential ansatz for D is found to be satisfactory for all z if $z_0 > 10$, but for smaller internal reflection we found less agreement. The line shape is approximately given by

$$I_c(\theta) \approx \frac{1}{1 - z_e(0)/\xi + z_e(0)\sqrt{q_\|^2 + 1/\xi^2}}\,. \tag{20}$$

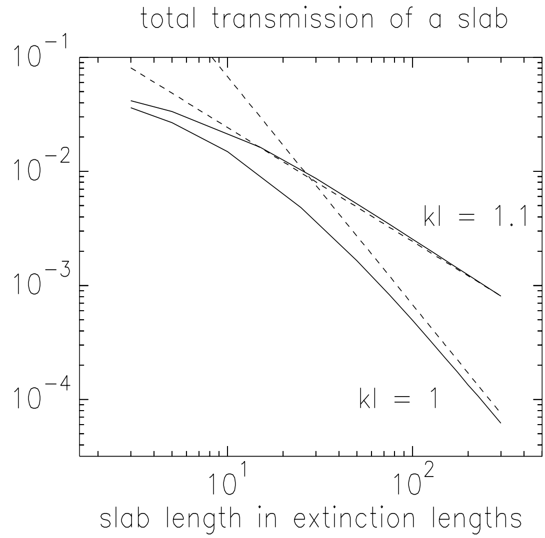

Figure 1. Numerical solution of the self-consistent equations for a finite slab. Total transmission coefficient as a function of the slab length, L, for the critical value $k\ell = 1$, and in the delocalized regime $k\ell = 1.1$. The dashed lines have slopes -2 and -1. We have adopted an internal-reflection parameter $z_0 = 10$.

This indicates an analytical rounding for $\theta < 1/k\xi$, reminiscent of an absorbing semi-infinite medium in the delocalized regime, a case that must be excluded experimentally [16].

One can use the solution $D_\infty(z)$ for the semi-infinite medium to estimate the length dependence of the total transmission $T(L)$ of a slab with length L. We expect that $D(z) \approx D_\infty(\frac{1}{2}L - |\frac{1}{2}L - z|)$ i.e., symmetric in the central plane $z = \frac{1}{2}L$. For a point source close to the boundary $z = 0$, the diffusion equation predicts,

$$T(L) = z_0\ell \left(2z_0\ell + \int_0^L \mathrm{dz}\, \frac{\mathrm{D_B}}{\mathrm{D(z)}} \right)^{-1} . \tag{21}$$

The integral is proportional to the "optical thickness" of the slab. Equation (21) gives,

$$T(L) \to \left\{ \begin{array}{ll} 4z_0\,(D_\infty(0)/D_B)\,(\xi_c/\ell) \times (\ell/L)^2 & k\ell = 1 \\ z_0\,(D_\infty(0)/D_B)\,(\ell/\xi) \times \exp(-L/\xi) & k\ell < 1 \end{array} \right. . \tag{22}$$

This scale dependence agrees with scaling theory [5, 47], but has large and precise prefactors: 2.6 for $k\ell = 1$ and $z_0 = \frac{2}{3}$, and increasing with z_0.

Figure 1 shows that $1/L^2$ law predicted at the mobility edge rapidly disappears in the delocalized regime $k\ell > 1$. It has been reported by Genack*et al.* [14] for microwaves and by Wiersma *et al.* [15] for GaAs samples.

4. Application to a Tube

A "quasi 1D" medium is characterized by a transverse surface $A < \ell^2$, so that only the transverse mode $q_{\parallel} = 0$ contributes to Eq. (13), with weight $1/A$. It differs from a genuine 1D medium in that the diffusion picture is still believed to apply for not too large lengths. The tube geometry is studied experimentally with microwaves in the group of Genack [48]. Define the length $\xi = A\rho(k)v_E\ell$. In RMT this length emerges as the localization length of the tube in the presence of time-reversal [49]. Upon introducing the "optical depth" as

$$\tau(z) \equiv \int_0^z \mathrm{dz}' \, \frac{\mathrm{D_B}}{\mathrm{D(z')}} \,, \tag{23}$$

Equation (12) reduces to the conventional diffusion equation,

$$-\partial_\tau^2 \, F(\tau, \tau') = \delta(\tau - \tau') \,, \tag{24}$$

whose solution is easily obtained using the radiative boundary conditions at $\tau(0) = 0$ and $\tau(L) = b$. The self-consistent Eq. (13) imposes that

$$\frac{\mathrm{d}\tau}{\mathrm{dz}} = 1 + \frac{1}{\xi} \, F(\tau, \tau) = 1 + \frac{1}{\xi} \frac{(\tau + z_0)(b + z_0 - \tau)}{b + 2z_0} \,. \tag{25}$$

For a semi-infinite quasi 1D medium ($b = \infty$) this equation has the exact solution

$$D(z) = \left[\frac{1}{D_B} + \frac{2z_0}{v_E \xi} \right]^{-1} \exp(-2z/\xi) \,. \tag{26}$$

For a finite length of the tube the total transmission, calculated from Eq. (21), is plotted in Figure 2. It compares very well to the solution of RMT published by Zirnbauer [49], in particular the cross-over from ℓ/L to $\exp(-L/\xi)$. The RMT result has an extra $1/L^{3/2}$ factor in the transmission. Not unexpectedly, Eq. (26) shows the diffusion coefficient at the boundary to be sensitive to the amount of internal reflection, as quantified by the extrapolation parameter z_0. This was also seen to be the case for the slab geometry.

5. Application to a Sphere: Thouless' Criterion

For a sphere with radius a one expects a diffusion coefficient $D(r)$ that depends only on the distance r to the origin. An expansion into spherical harmonics yields the self-consistent problem,

$$-\partial_r r^2 D(r) \partial_r F_l(r, r') + D(r) l(l+1) F_l(r, r') = \delta(r - r') \tag{27}$$

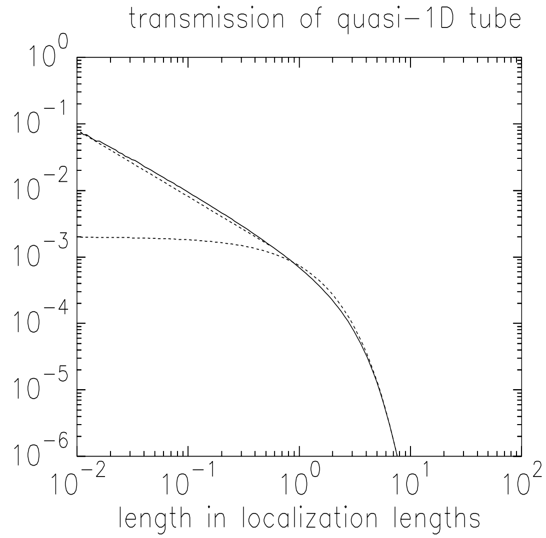

Figure 2. Solution of the self-consistent equations for a quasi-1D medium. Plotted is the average transmission as a function of the thickness of the tube. Dashed lines denote the Ohmic $1/L$ behavior that applies for small lengths and the localized exponential law that applies for large lengths. The extrapolation length has been chosen much smaller than the localization length.

$$\frac{1}{D(r)} = \frac{1}{D_B} + \frac{1}{k^2\ell}\sum_l (2l+1)\, F_l(r,r) \tag{28}$$

$$F_l(a,r') + \frac{3z_0 D(a)}{v_E}\partial_r F_l(a,r') = 0\,. \tag{29}$$

The sum over the angular quantum numbers l diverges. It is reasonable to adopt an upper cut-off $l_{\max} \approx r/\ell$ which is consistent with the choice $q_{\max} = 1/\ell$ made earlier. In particular, close to the center ($r < \ell$) only the s-wave spherical harmonic $l = 0$ contributes. The above equations will be solved for a very large sphere $a \gg \ell$ and for the critical value $k\ell = 1$. We will verify Thouless' assertion that the diffusion coefficient at the mobility edge scales inversely with the size of the sphere, i.e., $D \sim 1/a$ (see e.g., Ref. [31] for a good discussion).

Equation (27) can easily be solved analytically for $l = 0$. Inserting its solution into Eq. (28) yields,

$$\frac{1}{D(0)} = \frac{1}{D_B} + \frac{1}{k^2\ell}\left[\int_\ell^a \frac{\mathrm{dr}}{\mathrm{r^2 D(r)}} + \frac{z_0\ell}{v_E a^2}\right]\,. \tag{30}$$

For a very large sphere, the solution (17) for a planar slab should apply near the boundaries. At the mobility edge $k\ell = 1$ we assert the profile,

$$D(r) = D(a)\frac{1 + (a-r)/z_c}{1 + (a-r)/\xi_c}\,, \tag{31}$$

with $\xi_c \ll z_c \ll a$. The parameters ξ_c and $D(a)$ are known from the slab geometry. The length z_c is to be determined and denotes the depth beyond which the diffusion coefficient takes its bulk value. Putting the conjecture (31) into Eq. (30) gives $z_c \approx a\sqrt{\xi_c/\log(a/\ell)}\sqrt{D(a)/D_B}$. As a result, at depths exceeding z_c the diffusion profile takes the constant value

$$D \approx D(a)\frac{\xi_c}{z_c} \approx D(a) \times \frac{\xi_c}{a}\sqrt{\log(a/\ell)}\,. \tag{32}$$

This confirms the Thouless conjecture that at the mobility edge the diffusion coefficient D scales inversely with the size of the sphere. Note that, for our approach to be valid, we demand $\sqrt{\log(a/\ell)}$ to be a large number. The sphere must thus really be very large.

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